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Generalized Hybrid Method for Fuzzy Multiobjective Optimization of Engineering Systems

S. S. Rao* and Li Chen†
Purdue University, West Lafayette, Indiana 47907-1288

A generalized hybrid approach is presented for the multiobjective optimization of engineering systems in the presence of objectives and constraints that are partly fuzzy and partly crisp. The methodology is based on both fuzzy-set and Dempster–Shafer theories to capture the features of incomplete, imprecise, uncertain, or vague information that is often present in real-world engineering systems. The original partly fuzzy multiobjective optimization problem is first defuzzified into a crisp generalized multiobjective optimization problem using fuzzy-set theory. The resulting multiobjective problem is then transformed into an equivalent single-objective optimization problem using a modified Dempster–Shafer theory. The computational details of the approach are illustrated with a structural design example.

Nomenclature

A_α = α-level cut of the fuzzy set A
b_j = upper bound of the jth constraint
$\tilde{D}$ = fuzzy design solution domain
$\tilde{F}_i$ = fuzzy set of the ith objective function
$\vec{f}$ = vector of crisp objective functions
$\vec{f}'$ = generalized objective-function vector
f_i = ith crisp objective function
f_i' = ith generalized objective function
$\tilde{G}_j$ = fuzzy set of the jth constraint
g_j = jth crisp constraint
$m(\cdot)$ = basic probability assignment (bpa) function
$S(\cdot)$ = overall degree of belief (disbelief or ignorance)
S_f = satisfaction function
X = universe of discourse
$\boldsymbol{x}$ = design vector (or elementary proposition)
$\boldsymbol{x}'$ = generalized design vector
$\neg\boldsymbol{x}$ = complementary proposition of $\boldsymbol{x}$
α = design level cut
δ_j = leeway of the jth constraint
η, ζ = scalar variables
λ = scalar factor (parameter)
$\mu(\cdot)$ = membership function
2^X = power set of X
$\emptyset$ = empty set

Introduction

MOST engineering system design problems require the consideration of multiple and conflicting objectives more often than a single objective. The designer may need to infer evidential reasoning or make decisions on certain design schemes by collecting and combining relevant evidence (or experience) for or against some hypotheses of the design. For example, the design optimization of an engineering system can be considered as a decision-making process in which the designer (or computer in the case of computer-aided design) iteratively compares the previously computed results with the ones currently generated, and achieving the optimal design can be regarded as the final step of the decision-making process. A general multiobjective optimum-design problem can be expressed as follows:

Minimize

$$\vec{f}(\boldsymbol{x}) = \begin{bmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_k(\boldsymbol{x}) \end{bmatrix} \tag{1}$$

subject to

$$g_j(\boldsymbol{x}) \le b_j, \qquad j = 1, 2, 3, \ldots, l \tag{2}$$

where $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]^T$ is the design vector and the design criteria $f_j(\boldsymbol{x})$ may be conflicting and noncommensurable.

Insight into the nature of the optimization process leads to the conclusion that incomplete or imprecise information is often involved in the computational process although, simultaneously, there still exists a substantial amount of acquired computational information associated with the process. This is because at each iterative phase of the solution-seeking process, in most cases, the (selected) trial design vector, which serves as a potential candidate for optimum solution, often represents only the intermediate (or imprecise) and not the final (or precise) information. Moreover, in modeling real-world design problems, precise (deterministic or crisp) mathematical forms are always used to depict a problem for simplicity and convenience, rather than the real-world (linguistic) situation, which is often stated in an uncertain or fuzzy way. In engineering optimization, it may not be possible to describe the objectives precisely, because the utility functions are not definable precisely or the phenomena of the design problem can only be stated in an ambiguous way; it may also not be possible to state the constraints with certainty, because doubt may arise about the exactness of permissible or bound values, degree of credibility, and correctness of statements and judgments on the constraints. In other words, real-world engineering design problems often suffer not only from incompleteness, but also from uncertainty, inexactness, or vagueness.

To handle the problems described above, a novel approach is presented in this work. This approach serves as a bridge that brings together the Dempster–Shafer (DS) theory and the fuzzy-set theory in a hybrid fashion so that both incomplete and vague information can be handled and manipulated in the design optimization process. The DS theory permits the use of probabilistic judgments to capture the incomplete nature of the evidence based on probability theory,[1–3] and recently it has been applied to knowledge-based engineering systems.[4,5] The fuzzy set theory is well known to represent and manage vague information, and has been extensively studied and applied to the area of engineering optimization.[6–10]

The following sections introduce two approaches for specific cases in which the whole (not partial) set of objectives is considered either fuzzy or crisp. A novel hybrid method is then proposed

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*Professor, School of Mechanical Engineering. Member AIAA.

†Graduate Research Assistant, School of Mechanical Engineering.

for solving general multiobjective optimization problems in which the objectives and constraints are partly crisp and partly fuzzy. Finally, a numerical example of structural optimization is presented to illustrate the computational aspects of the methodology.

Fuzzy Model for Specific Cases

The conventional optimization approaches to design assume that the design data required or acquired are known precisely, that the design constraints delimit a well-defined set of feasible decisions, and that the design objectives are well defined and easy to formulate. In a real-world optimization environment, some of the mathematical expressions (equations) used to represent the design criteria (objectives) and constraints may be empirical, and some of the data used in the design process may be obtained from a limited set of experiments. This implies that the empirical formulas and experimental data used may involve uncertainty, incompleteness, and imprecision because of our imperfect understanding of the events or, possibly, imperfect measurement of the intensity of the events. For example, a crisp set of lower- and upper-bound values is usually imposed on design variables in an optimization process. Under some circumstances, the boundary values may not be known for sure. In such cases, there should be a permissible leeway for the design variables (or constraints) over the stated transition zone.

To realistically depict many real-world design problems, the notion of optimization needs to be modified using linguistic mathematical terms via fuzzy-set theory. It is realized that both the fuzzy objective functions and the fuzzy constraints are characterized by their membership functions. Since the overall optimization process requires a simultaneous satisfaction of the objective functions and the constraints, a decision or selection of a set of design variables is reached by assuming that the constraints are noninteractive and that the logical and (min) operator corresponds to a fuzzy intersection. Two fuzzy models of multiobjective optimization are presented—one for problems with fuzziness in constraints only, and the other for problems with fuzziness in both constraints and objectives.

Fuzzy Model 1: Some constraints are fuzzy. Consider a multiple objective optimization problem with l constraints among which only p constraints $g_j,\ j = 1, \ldots, p$, are crisp, and the others are fuzzy, represented by the fuzzy (constraint) sets $\bar{G}_j,\ j = p+1, \ldots, l$. If only the constraints (but not the objective functions) are fuzzy, then the fuzzy optimization problem can be solved using the level-cut (or α-cut) approach.[6] The basic concepts of fuzzy set theory are summarized in Appendix A. The α cut A_α of a fuzzy set A is defined as

$$A_\alpha = \{x \in X,\ \mu_A \geq \alpha\} \tag{3}$$

where X is a universe of discourse of the element x, and μ_A is defined as a membership function of the fuzzy set A. In the level-cut method, an optimal design vector is considered to be a solution that has at least a certain degree of membership in the fuzzy domain $\bar{D}$ defined as

$$\bar{D} = \bar{G}_{p+1} \cap \bar{G}_{p+2} \cap \cdots \cap \bar{G}_l \tag{4}$$

A fuzzy decision is made at the congregation of the total of $l - p$ fuzzy constraints in terms of membership values, given by

$$\mu_{\bar{D}}(\boldsymbol{x}) = \bigcap_{j=p+1}^{l} \mu_{\bar{G}_j}[g_j(\boldsymbol{x})] \geq \alpha \tag{5}$$

with design level α. For a single-objective optimization problem, a sequence of level optimum solutions may be found with different α values ($\alpha_1 \leq \alpha_2 \leq \cdots$) that indicate the various optimum level design schemes. In the multiobjective optimization process, an optimum design level, denoted as α^*, can be determined uniquely among the level optimum schemes corresponding to the various objective functions such that the individual objectives reach, simultaneously, a common design scheme that represents the best compromise solution over the feasible design space. Furthermore, the fuzzy optimization problem can be transformed into a crisp (nonfuzzy) optimization problem as follows: For a prescribed design level α ($0 \leq \alpha \leq 1$), find the design vector $\boldsymbol{x}$

to minimize

$$\vec{f}(\boldsymbol{x}) = \begin{bmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_k(\boldsymbol{x}) \end{bmatrix} \tag{6}$$

subject to

$$\begin{aligned} g_j(\boldsymbol{x}) &\leq b_j, & j &= 1, 2, \ldots, p \\ \alpha &\leq \mu_{\bar{G}_j}(\boldsymbol{x}), & j &= p+1, \ldots, l \end{aligned} \tag{7}$$

where $\bar{G}_j$ denotes the admissible tolerance interval for the jth fuzzy constraint, and a bar over a symbol implies that the expression or variable contains fuzzy information.

Fuzzy Model 2: Some constraints, and all *objectives are fuzzy.* Consider a multiobjective optimization problem with k fuzzy objectives $f_i, i = 1, \ldots, k$, represented by the fuzzy sets $\bar{F}_i,\ i = 1, \ldots, k$, and $l - p$ fuzzy constraints $g_j,\ j = p+1, \ldots, l$, denoted by the fuzzy sets $\bar{G}_j,\ j = p+1, \ldots, l$. A fuzzy decision is defined as the intersection of the fuzzy objectives and the fuzzy constraints as[7,8]

$$\bar{D} = (\bar{F}_1 \cap \bar{F}_2 \cap \cdots \cap \bar{F}_k) \cap (\bar{G}_{p+1} \cap \bar{G}_{p+2} \cap \cdots \cap \bar{G}_l) \tag{8}$$

and in terms of membership values as

$$\mu_{\bar{D}}(\boldsymbol{x}) = \left[\bigcap_{i=l}^{k} \mu_{\bar{F}_i}(\boldsymbol{x})\right] \cap \left\{\bigcap_{j=p+1}^{l} \mu_{\bar{G}_j}[g_j(\boldsymbol{x})]\right\} \tag{9}$$

The optimum solution $\boldsymbol{x}^*$ is then given by

$$\mu_{\bar{D}}(\boldsymbol{x}^*) = \max\ \mu_{\bar{D}}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \bar{D} \tag{10}$$

with

$$\mu_{\bar{D}}(\boldsymbol{x}) = \min_{i,j} \left\{\mu_{\bar{F}_i}(\boldsymbol{x}),\ \mu_{\bar{G}_j}(\boldsymbol{x})\right\} \tag{11}$$

Thus the multiobjective optimization problem is converted into an equivalent single objective problem as follows: Find a generalized design vector

$$\boldsymbol{x}' = [\boldsymbol{x},\ \eta]^T \tag{12}$$

to maximize

$$\eta \tag{13}$$

subject to

$$\begin{aligned} g_j(\boldsymbol{x}) &\leq b_j, & j &= 1, 2, \ldots, p \\ \eta &\leq \mu_{\bar{G}_j}(\boldsymbol{x}), & j &= p+1, \ldots, l \\ \eta &\leq \mu_{\bar{F}_i}(\boldsymbol{x}), & i &= 1, 2, \ldots, k \end{aligned} \tag{14}$$

Fuzzy Model for a General Case

The fuzzy models discussed in the preceding section require all objectives to be either crisp or fuzzy. A more realistic model is developed in this section for the solution of a general fuzzy multiobjective optimization problem, in which the objectives and/or the constraints are partly fuzzy and partly crisp. Consider a multiobjective optimization problem with a total of k objectives, in which q criteria $f_i,\ i = 1, \ldots, q$, are crisp and the other $k - q$ criteria $f_i,\ i = q+1, \ldots, k$, are fuzzy, being represented by the fuzzy sets $\bar{F}_i,\ i = q+1, \ldots, k$; there also exist a total of l constraints, of which the first p constraints $g_j,\ j = 1, \ldots, p$, are crisp, and the others are fuzzy, represented by the fuzzy sets $\bar{G}_j,\ j = p+1, \ldots, l$. The fuzzy domain $\bar{D}$ in which a fuzzy decision (design vector) is reached over the fuzzy feasible region is then defined as

$$\bar{D} = (\bar{F}_{q+1} \cap \bar{F}_{q+2} \cap \cdots \cap \bar{F}_k) \cap (\bar{G}_{p+1} \cap \bar{G}_{p+2} \cap \cdots \cap \bar{G}_l) \tag{15}$$

and a (fuzzy) design vector x is considered feasible and selected in terms of membership values as

$$\mu_{\tilde{D}}(x) = \left[\bigcap_{i=1}^{k} \mu_{\tilde{F}_i}(x)\right] \cap \left\{\bigcap_{j=p+1}^{l} \mu_{\tilde{G}_j}[g_j(x)]\right\} \quad (16)$$

An optimum design vector x^* is found by complying with

$$\mu_{\tilde{D}}(x^*) = \max\ \mu_{\tilde{D}}(x)$$
$$= \max\left\{\min_{i,j}[\mu_{\tilde{F}_i}(x), \mu_{\tilde{G}_j}(x)]\right\}, \qquad x \in \tilde{D} \quad (17)$$

Thus the fuzzy multiobjective optimization problem can be transformed into an ordinary nonfuzzy multiobjective optimization problem as follows: Find a generalized design vector

$$x' = [x, \zeta]^T \quad (18)$$

that minimizes

$$f'(x') = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_q \\ -\zeta \end{bmatrix} \quad (19)$$

subject to

$$\begin{aligned} g_j(x) &\le b_j, & j &= 1, 2, \ldots, p \\ \zeta &\le \mu_{\tilde{G}_j}(x), & j &= p+1, \ldots, l \\ \zeta &\le \mu_{\tilde{F}_i}(x), & i &= q+1, \ldots, k \end{aligned} \quad (20)$$

where $x = [x_1, x_2, \ldots, x_n]^T$ is the design vector, $f'(x')$ is the generalized objective vector, and ζ is maximized by minimizing $-\zeta$.

Optimization Using DS Theory

For a general fuzzy model stated in Eqs. (18–20), the central issue is how to seek the best compromise solution between the (possibly) mutually conflicting and often noncommensurable design criteria over the feasible region bounded by the design constraints. A recent method, presented by the authors, defines a satisfaction function based on the DS theory[11] that permits the transformation of an original multiobjective design problem into an ordinary single-objective optimization problem so that standard optimization techniques can be used to solve the problem. The methodology of implementing the DS-theory-based optimization is highlighted below. The basic concepts of DS theory are outlined in Appendix B for completeness.

In the design optimization environment, each design objective can be treated as an independent knowledge source; and the concept of universe of discourse, X, can be extended to define an arbitrary solution (design vector) selected from the feasible domain being either optimal or nonoptimal:

$$X = [x, \neg x : x \in g_j(x),\ j = 1, 2, \ldots, l] \quad (21)$$

where x represents the elementary proposition the selected vector x is an optimum solution, and $\neg x$ indicates the elementary proposition the selected vector x is not an optimum solution. Thus the power set of X, used to denote all possible propositions, can be expressed as

$$2^X = (\emptyset, x, \neg x, X) \quad (22)$$

where the element X denotes the proposition the selected vector x being an optimal solution could be either true or false, but is not sure, and $\emptyset$ indicates an empty set that is not of interest to us. During the iterative process of a solution-seeking procedure, the trial vector selected at any stage serves as a potential candidate for the optimum solution; it can be considered to represent incomplete data that provide only an intermediate information. Thus the vague or imprecise information, available for determining whether or not the selected vector x is an optimum solution, can be represented through a belief structure

$$[m(x), m(\neg x), m(X)] \quad (23)$$

where

$$m(x) + m(\neg x) + m(X) = 1 \quad (24)$$

Note that $m(\emptyset) = 0$ in the DS theory. Thus the method of finding the optimum solution can be depicted as a procedure in which the designer first accumulates computational information through a search process in the design space, then takes a comprehensive view of the design criteria, and finally makes a decision on the best compromise solution (based on the cumulative evidence acquired) among all the available design vectors.

Two steps, one based on Dempster's rule and the other based on Yager's rule, are used to incorporate the computational evidence (information) during the search for the best compromise solution. Dempster's rule is used to merge the belief structures given by the computational information for a single design criterion, and Yager's rule is used to evaluate a comprehensive satisfaction function for the multiple design criteria by treating them as independent knowledge sources. The detailed procedure is explained in the following:

The DS theory is based on available information and relies on evidence-based decision making under one or more independent knowledge sources. In searching for the optimum solution of a single-criterion problem, one usually selects a design vector x in the feasible region and calculates the corresponding objective-function value in each iteration. After comparing the current objective-function value with the previous one, the vector corresponding to the smaller objective function value is retained. This search process is repeated until the optimum solution is found over the feasible domain. The search for an optimum solution can be viewed as a decision-making process based on the computational information and evidence accumulated. It can also be seen that the various design criteria, measured at any stage of the computational procedure, provide different sources of information and evidence for drawing inferences about the best compromise solution.

Suppose that evidence is accumulated through the computational procedure of a single-criterion optimization. The independent sources of evidence for a single objective are assumed to constitute a set of two original belief structures, which are simple support functions on proposition x and its negation $\neg x$ given by

$$[m_{i,1}(\neg x), m_{i,1}(X)] \quad (25)$$

and

$$[m_{i,2}(x), m_{i,2}(X)] \quad (26)$$

which satisfy the relations

$$m_{i,1}(\neg x) + m_{i,1}(X) = 1 \quad (27)$$

and

$$m_{i,2}(x) + m_{i,2}(X) = 1 \quad (28)$$

where i indicates the original belief structures corresponding to the ith design criterion.

The computed information is used to construct basic probability assignment (bpa) functions for quantitatively determining the degrees of belief for each individual objective function. A belief function can be considered an extension of the membership function used in fuzzy-set theory.[3] In the absence of additional information, a linear variation[6,7] is assumed to define the bpa functions. Although no computational experience exists about nonlinear bpa functions, linear membership functions are adequate for most situations in fuzzy set theory.[8] In computing the bpa functions, $m_{i,1}(\neg x)$ denotes the degree of belief indicating the extent to which the selected vector x is not a candidate for the optimum solution; $m_{i,2}(x)$ denotes the degree of credibility indicating the closeness of the selected vector x to the optimum solution, and $m_{i,1}(X)$ and $m_{i,2}(X)$ denote the degrees of ignorance or vagueness in that the selected vector x is neither clearly accepted nor positively rejected as a candidate for the optimum solution.

Since there exist two original belief structures for any objective function, Dempster's rule is used to combine them into a single belief structure. By employing Eq. (B11), the result can be expressed as

$$m_i(x) = \frac{m_{i,1}(X)m_{i,2}(x)}{1 - m_{i,1}(\neg x)m_{i,2}(x)} \tag{29}$$

$$m_i(\neg x) = \frac{m_{i,2}(X)m_{i,1}(\neg x)}{1 - m_{i,1}(\neg x)m_{i,2}(x)} \tag{30}$$

$$m_i(X) = \frac{m_{i,1}(X)m_{i,2}(X)}{1 - m_{i,1}(\neg x)m_{i,2}(x)} \tag{31}$$

for $i = 1, 2, \ldots, k$. It can be easily verified that $m_i(x) + m_i(\neg x) + m_i(X) = 1$. Thus the three bpa functions associated with the comprehensive belief structure of the ith design criterion are given by Eqs. (29–31), where $m_i(x)$, $m_i(\neg x)$, and $m_i(X)$ represent, respectively, the degrees of belief, disbelief, and ignorance (or vagueness) in the proposition the selected vector x is a candidate for the optimum solution of the ith objective function.

The computational evidence and the information gathered for any objective function support the overall belief structure

$$\{m_i(x), m_i(\neg x), m_i(X) : 1 \le i \le k\} \tag{32}$$

for all the design criteria. All the design criteria ($i = 1, 2, \ldots, k$) are treated equally in the presence of a set of independent knowledge sources KS_j ($j = 1, 2, \ldots, k$) in DS theory. This leads to the problem of how the belief structures given by the various conflicting design criteria are to be used in a comprehensive manner. Yager's rule provides a basis for combining the various belief structures in which some of the independent knowledge sources might be conflicting or contradictory to a certain extent.

Recalling Dempster's rule formulated as Eq. (B11), it is easy to see that the equation corresponding to the combined bpa function $m(C)$ is homogeneous. The result of the knowledge-combining (or evidence-combining) process given by Dempster's rule depends on the individual input (evidence) values but not on the order in which they are combined. Equation (B12), associated with Yager's rule for incorporating the various bpa functions, however, is nonhomogeneous because of the presence of the term $m_1(\emptyset)m_2(\emptyset)$. Thus, the knowledge-merging (evidence-merging) process is sensitive to the order of combination when the number of independent knowledge sources (or design criteria) is greater than or equal to 3 ($k \ge 3$). To decouple the nonhomogeneous effect associated with the order of combination in Yager's rule, the individual sets of belief structure given by the various design criteria are arranged in descending order of their values of $m_i(x)$, $i = 1, 2, \ldots, k$ ($k \ge 3$). Yager's rule is then applied for combining the belief structures of the various design criteria into an overall one in the presence of multiple design criteria.

Starting with Eq. (B12), a recursive procedure can be developed for merging the belief structures of all the design criteria as

$$S_{i+1}(x) = S_i(x)m_{i+1}(x) + S_i(x)m_{i+1}(X) + S_i(X)m_{i+1}(x) \tag{33}$$

$$S_{i+1}(\neg x) = S_i(\neg x)m_{i+1}(\neg x) + S_i(\neg x)m_{i+1}(X) + S_i(X)m_{i+1}(\neg x) \tag{34}$$

$$S_{i+1}(X) = S_i(X)m_{i+1}(X) + S_i(\neg x)m_{i+1}(x) + S_i(x)m_{i+1}(\neg x), \qquad i = 1, 2, \ldots, k-1 \tag{35}$$

with

$$S_1(x) = m_1(x), \qquad S_1(\neg x) = m_1(\neg x) \tag{36}$$

$$S_1(X) = m_1(X)$$

where $S_k(x)$, $S_k(\neg x)$, and $S_k(X)$ indicate, respectively, the overall degrees of belief, disbelief, and ignorance (or vagueness) associated with the proposition the selected vector x is an optimum solution for all the design criteria. Since the belief structures, at each stage, combine two design criteria, the total number of recursive steps is $k - 1$. It is also easy to see that $S_i(x) + S_i(\neg x) + S_i(X) = 1$ for $i = 1, 2, \ldots, k - 1$. An overall satisfaction function, $S_f(x)$, is then defined as

$$S_f(x) = S_k(x), \qquad 0 \le S_f(x) \le 1 \tag{37}$$

Thus the multiobjective optimum design problem can be viewed as an equivalent single-objective optimization problem: Find a design vector x in the feasible domain that maximizes the satisfaction function $S_f(x)$ given by Eq. (37). The resulting single-objective optimization problem can be solved using any of the standard solution techniques.[12]

Computational Procedure

The best compromise solution to the general fuzzy multiobjective problem using the proposed generalized hybrid approach can be determined using the following step-by-step procedure.

1) Find the solution of each individual single-objective optimization problem by defuzzifying the fuzzy objective into a crisp (nonfuzzy) form.

2) Determine the best (or minimum) and a possible worst (or maximum) solution of each objective function that is defined as fuzzy in the original problem statement.

3) Use the solutions determined in step 2 as the boundaries of the fuzzy range in the corresponding fuzzy optimization problem.

4) Transform the fuzzy multiobjective problem into a generalized crisp multiobjective problem.

5) Construct the two original belief structures for each objective function.

6) Convert the generalized multicriterion optimization problem into a corresponding single optimization problem by defining the satisfaction function.

7) Solve the resulting crisp (ordinary) single-objective optimization problem using any of the established techniques.[12]

As stated earlier, linear membership functions are assumed in this work for simplicity.[3,6,7] The membership function of a fuzzy objective function (criterion) is constructed as

$$\mu_{\tilde{F}_i}(x) = \begin{cases} 0 & \text{if } f_i(x) > f_i^{\max} \\ \dfrac{f_i^{\max} - f_i(x)}{f_i^{\max} - f_i^{\min}} & \text{if } f_i^{\min} < f_i(x) \le f_i^{\max} \\ & \quad (i = q+1, \ldots, k) \\ 1 & \text{if } f_i(x) \le f_i^{\min} \end{cases} \tag{38}$$

where $f_i^{\min} = \min_j f_i(x_j^*) = f_i(x_i^*)$, $f_i^{\max} = \max_j f_i(x_j^*)$, and x_j^* is the optimum design vector of the jth objective function ($j = q + 1, \ldots, k$). If the fuzzy constraints are considered as

$$g_j(x) \le b_j + \delta_j, \qquad j = p + 1, \ldots, l \tag{39}$$

where δ_j denotes the distance by which the boundary of the jth constraint is moved, then a membership function corresponding to the jth constraint is defined as

$$\mu_{\tilde{G}_j}(x) = \begin{cases} 0 & \text{if } g_j(x) \ge b_j + \delta_j \\ 1 - \dfrac{g_j(x) - b_j}{\delta_j} & \text{if } b_j < g_j(x) < b_j + \delta_j \\ & \quad (j = p+1, \ldots, l) \\ 1 & \text{if } g_j(x) \le b_j \end{cases} \tag{40}$$

Once the membership functions have been established for the fuzzy objectives and constraints, the original fuzzy multiobjective optimization problem can be converted into an equivalent ordinary (crisp) multiobjective optimization problem as stated in Eqs. (18–20). Then the modified DS theory approach can be used for its solution.

As stated earlier, the DS-theory approach for multiobjective optimization starts with the construction of two-original belief structures for each objective. Let evidence be accumulated during the computational process of single-criterion optimization. Let the computations yield an evidence (estimate) of the minimum ($f_i^{\prime\min}$) and two upper bounds on the maximum ($f_{i,2}^{\prime\max}$ and $f_{i,1}^{\prime\max}$) of the ith generalized objective function f_i':

$$f_i^{\min} = \min_j f_i\left(x_j^*\right) = f_i'\left(x_i^*\right) \tag{41}$$

$$f_{i,2}^{\prime\max} = \max_j f_i'\left(x_j^{\prime *}\right) \tag{42}$$

$$f_{i,1}^{\prime\max} = \begin{cases} \lambda f_{i,2}^{\prime\max} & \text{if } f_{i,2}^{\prime\max} \geq 0 \\ (1/\lambda) f_{i,2}^{\prime\max} & \text{if } f_{i,2}^{\prime\max} < 0 \end{cases} \tag{43}$$

with

$$\lambda > 1, \qquad 1 \leq i \leq q+1, \qquad \text{and} \qquad 1 \leq j \leq q+1$$

where $x' = [x, \zeta]^T$ denotes the generalized design vector, and $q+1$ is the number of design criteria; $x_j^{\prime *}$ is an optimum solution of the jth generalized criterion, and λ is a factor used to adjust the first upper bound on the maximum value more closely to the true maximum value (usually, λ will have a value between 1 and 10).

The computational information is used to construct the bpa functions for determining the degrees of belief corresponding to the minimum and the upper bounds on the maximum of each objective function. A belief function can be considered as an extension of the membership function used in the fuzzy-set theory.[3] A linear bpa function is given by

$$m_{i,1}(\neg x') = \begin{cases} 0 & \text{if } f_i'(x') < f_i^{\prime\min} \\ \dfrac{f_i(x') - f_i^{\prime\min}}{f_{i,1}^{\prime\max} - f_i^{\prime\min}} & \text{if } f_i^{\prime\min} < f_i'(x') \leq f_{i,1}^{\prime\max} \\ 1 & \text{if } f_i'(x') \geq f_{i,1}^{\prime\max} \end{cases} \tag{44}$$

and

$$m_{i,2}(x') = \begin{cases} 0 & \text{if } f_i'(x') > f_{i,2}^{\prime\max} \\ \dfrac{f_{i,2}^{\prime\max} - f_i'(x')}{f_{i,2}^{\prime\max} - f_i^{\prime\min}} & \text{if } f_i^{\prime\min} < f_i'(x') \leq f_{i,2}^{\prime\max} \\ 1 & \text{if } f_i'(x') \leq f_i^{\prime\min} \end{cases} \tag{45}$$

Note that $m_{i,1}(X)$ and $m_{i,2}(X)$ can be determined using Eqs. (27) and (28). Equations (44) and (45) can be interpreted as follows. The vector x', found at any step of the interative process, is assigned a numerical value between 0 and 1 representing the computational evidence observed through the minimum and the upper bounds on the maximum of the particular (single) objective function. Since the inequality $f_{i,2}^{\prime\max} - f_i^{\prime\min} < f_{i,1}^{\prime\max} - f_i^{\prime\min}$ always holds true, it is appropriate to associate the smaller difference with the bpa function corresponding to the proposition x', and the larger one with the one corresponding to the proposition $\neg x'$.

The two original belief structures corresponding to any particular generalized objective function are to be merged into a single belief structure associated with the current design using Eqs. (29–31). This yields the degrees of belief $m_i(x')$, disbelief $m_i(\neg x')$, and ignorance $m_i(X)$ for the ith criterion at the current generalized design vector x'. The individual sets of belief structures are then arranged in descending order of the values of $m_i(x')$, $i = 1, 2, \ldots, q+1$. Thus, all the belief structures of the individual criteria at the current design vector x' are comprehensively fused into a single belief structure by applying the recursive formulas (33–36). The overall satisfaction function corresponding to the current design vector x' is then identified using Eq. (37). Finally, the original fuzzy multicriterion optimization problem is stated as an equivalent crisp single-objective optimization problem as follows:

Maximize

$$S_f(x') = \max_{x \in G'}[S_k(x')] \tag{46}$$

where G' denotes the feasible domain bounded by the constraints of Eq. (20). The problem stated in Eq. (46) is then solved using any of the standard mathematical programming techniques.

Illustrative Example

The methodology developed in this work is illustrated through a structural design example to indicate the computational details of the approach. The example is a 25-bar truss, shown in Fig. 1, that is required to support the two load conditions given in Table 1. Three design criteria are considered: the minimization of weight, the minimization of the deflections of nodes 1 and 2, and the maximization of the fundamental natural frequency of vibration of the truss. Two of the design criteria (the weight and the deflection) are considered fuzzy. The constraints on the problem include yielding and Euler buckling of members. The members are assumed to be tubular with cross-sectional areas allowed to vary from 0.1 to 5.0 in.^{2}. The permissible stresses for all the members are specified as 40,000 psi in both tension and compression. Leeways of 4,000 psi and 0.01 in.^{2} are allowed for the constraints on the member stresses and the lower bound of the cross-sectional areas, respectively. The Young's modulus and the material density are taken as $E = 10^7$ psi and $\rho = 0.1$ lb/in.^{3}. The members are assumed to have a nominal diameter-to-thickness ratio of 100, so that the buckling stress becomes[7]

$$p_i = \frac{-100.01\pi E A_i}{8 l_i^2}, \qquad i = 1, 2, \ldots, 25 \tag{47}$$

where A_i and l_i indicate, respectively, the cross-sectional area and length of member i. The member areas are grouped as follows:

$$A_1, \qquad A_2 = A_3 = A_4 = A_5, \qquad A_6 = A_7 = A_8 = A_9$$

$$A_{10} = A_{11}, \qquad A_{12} = A_{13}, \qquad A_{14} = A_{15} = A_{16} = A_{17}$$

$$A_{18} = A_{19} = A_{20} = A_{21}, \qquad A_{22} = A_{23} = A_{24} = A_{25}$$

Thus a total of eight independent areas are chosen as design variables. The problem can be stated as follows:

Find the design vector

$$x = \begin{bmatrix} x_1 = A_1 \\ x_2 = A_2 \\ x_3 = A_6 \\ x_4 = A_{10} \\ x_5 = A_{12} \\ x_6 = A_{14} \\ x_7 = A_{18} \\ x_8 = A_{22} \end{bmatrix} \tag{48}$$

that minimizes

weight:
$$f_1(x) = \sum_{i=1}^{25} \rho A_i l_i \tag{49}$$

deflection:
$$f_2(x) = \left(u_{1x}^2 + u_{2x}^2 + u_{1z}^2\right)^{\frac{1}{2}} + \left(u_{2x}^2 + u_{2y}^2 + u_{2z}^2\right)^{\frac{1}{2}} \tag{50}$$

frequency:
$$f_3(x) = -\omega_1 \tag{51}$$

Table 1 Loads acting on the 25-bar truss

Load condition	Component joint	Load, lb			
		1	2	3	6
1	x	0	0	0	0
	y	20,000	−20,000	0	0
	z	−5,000	−5,000	0	0
2	x	1,000	0	500	500
	y	10,000	10,000	0	0
	z	−5,000	−5,000	0	0

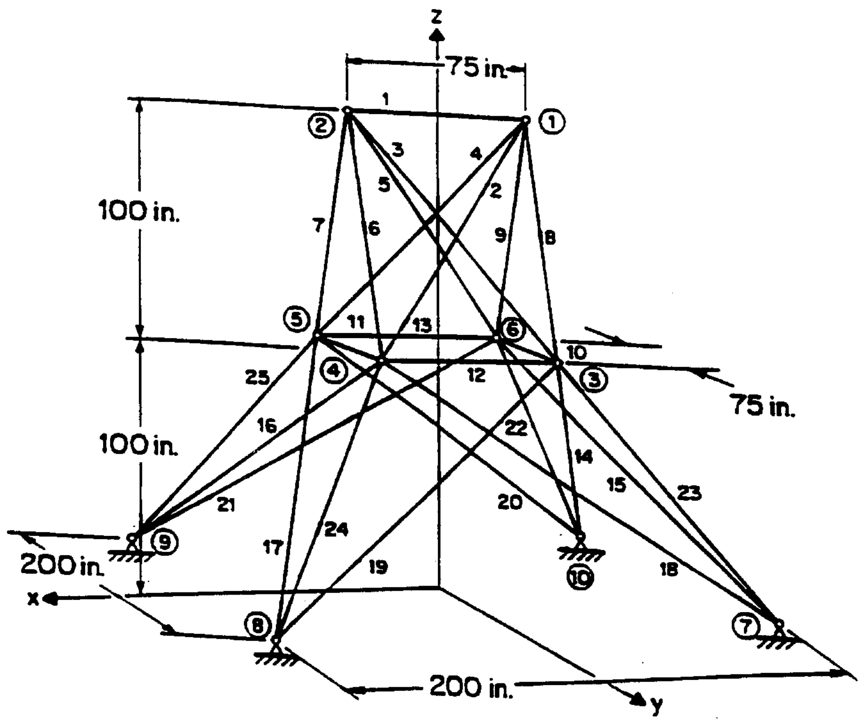

Fig. 1 25-bar truss.

subject to

$$|\sigma_{ij}(x)| \le 40{,}000 + \max(4000), \quad i = 1, 2, \ldots, 25, \quad j = 1, 2$$

$$\sigma_{ij}(x) \ge p_i(x), \quad i = 1, 2, \ldots, 25, \quad j = 1, 2 \tag{52}$$

$$0.1 - \max(0.01) \le x_i \le 5.0 \quad i = 1, 2, \ldots, 8$$

where u_{ix}, u_{iy}, and u_{iz} denote the x, y, z components of the deflection of node i ($i = 1, 2$), ω_1 denotes the fundamental natural frequency of vibration, and σ_{ij} denotes the stress in member i in load condition j.

The single criterion optimization problems are first solved to obtain the best (optimal) and worst (upper bound) solutions of the objective functions that are considered fuzzy. Based on the numerical results given in Table 2, linear membership functions for the fuzzy objectives and constraints are constructed as

$$\mu_{\tilde{F}_1}(x) = \begin{cases} 0 & \text{if } f_1(x) > 1619.3258 \\ \dfrac{1619.3258 - f_1(x)}{1619.3258 - 233.0727} & \text{if } 233.0727 < f_1(x) \le 1619.3258 \\ 1 & \text{if } f_1(x) \le 233.0727 \end{cases} \tag{53}$$

$$\mu_{\tilde{F}_2}(x) = \begin{cases} 0 & \text{if } f_2(x) > 1.9250 \\ \dfrac{1.9250 - f_2(x)}{1.9250 - 0.3083} & \text{if } 0.3083 < f_2(x) \le 1.9250 \\ 1 & \text{if } f_2(x) \le 0.3083 \end{cases} \tag{54}$$

$$\mu_{|\tilde{\sigma}_{ij}|} = \begin{cases} 0 & \text{if } |\sigma_{ij}| \ge 44{,}000 \\ 1 - \dfrac{|\sigma_{ij}| - 40{,}000}{4000} & \text{if } 40{,}000 < |\sigma_{ij}| < 44{,}000 \\ 1 & \text{if } |\sigma_{ij}| \le 40{,}000 \end{cases} \tag{55}$$

$$\mu_{\tilde{x}_i} = \begin{cases} 0 & \text{if } x_i \le 0.09 \\ 1 - \dfrac{0.1 - x_i}{0.01} & \text{if } 0.09 < x_i < 0.1 \\ 1 & \text{if } x_i \ge 0.1 \end{cases} \tag{56}$$

Thus, the fuzzy multicriterion optimization problem can be converted into an equivalent generalized crisp multiobjective optimization problem as follows:

Find a generalized design vector

$$x' = [x, \zeta]^T \tag{57}$$

that minimizes the generalized objectives

$$\vec{f}'(x') = \begin{bmatrix} -\omega_1 \\ -\zeta \end{bmatrix} \tag{58}$$

such that

$$\sigma_{ij}(x) \ge p_i(x), \quad x_i \le 5.0, \quad \zeta \le \mu_{|\tilde{\sigma}_{ij}|}(x)$$
$$\zeta \le \mu_{\tilde{x}_i}(x), \quad i = 1, 2, \ldots, 25, \quad j = 1, 2 \tag{59}$$

where $f_1' = -\omega_1$ and $f_2' = -\zeta$ are the generalized objective functions. The minimum and two upper bounds on the maximum of the two generalized criteria (with $\lambda = 1.10$) are then identified, with results summarized in Table 2. The two sets of original belief structures can be defined for each of the two generalized criteria as

$$m_{1,1}(\neg x') = \begin{cases} 0 & \text{if } f_1'(x') < -108.6224 \\ \dfrac{f_1'(x') + 111.882}{-57.401 + 111.882} & \text{if } -108.6224 < f_1'(x') \le -57.401 \\ 1 & \text{if } f_1'(x') \ge -57.401 \end{cases} \tag{60a}$$

$$m_{1,1}(X) = 1 - m_{1,1}(\neg x') \tag{60b}$$

Table 2 Results of the design example

Design attributes	Single-obj. optimization min f_1 (weight)	min f_2 (deflection)	Generalized multiobj. optimization min f_1' $(= -\omega_1)$	min f_2' $(= -\zeta)$	Multiobj. optim. solution
X_1	0.1	3.7931	0.0935813	2.23962	0.116195
X_2	0.80228	5.0	0.756605	2.13473	1.11077
X_3	0.74789	5.0	0.818486	2.61012	1.32307
X_4	0.1	3.3183	4.01880	2.06411	2.13726
X_5	0.12452	5.0	4.15683	2.06411	2.18071
X_6	0.57117	5.0	4.26475	0.865886	2.17323
X_7	0.97851	5.0	3.39709	1.69237	2.19678
X_8	0.80247	5.0	4.99812	2.14397	4.68667
f_1 (weight)	233.07265	1619.3258	1019.76	600.807	746.985
f_2 (deflection)	1.924989	0.30834	1.22999	0.739053	0.907819
f_1' $(= -\omega_1)$	−73.25348	−70.2082	−111.882	−63.1410	−102.0285
f_2' $(= -\zeta)$	——	——	−0.262230	−0.733757	−0.629180
1st max bound	——	——	−57.40091	−0.238391	——
S_f	——	——	——	——	0.863086

$$m_{1,2}(x') = \begin{cases} 0 & \text{if } f_1'(x') > -63.141 \\ \dfrac{-63.141 - f_1'(x')}{-63.141 + 111.882} & \text{if } -111.882 < f_1'(x') \le -63.141 \\ 1 & \text{if } f_1'(x') \le -111.882 \end{cases} \tag{61a}$$

$$m_{1,2}(X) = 1 - m_{1,2}(x') \tag{61b}$$

and

$$m_{2,1}(\neg x') = \begin{cases} 0 & \text{if } f_2'(x') < -0.7338 \\ \dfrac{f_2'(x') + 0.7338}{-0.2384 + 0.7338} & \text{if } -0.7338 < f_2'(x') \le -0.2384 \\ 1 & \text{if } f_2'(x') \ge -0.2384 \end{cases} \tag{62a}$$

$$m_{2,1}(X) = 1 - m_{2,1}(\neg x') \tag{62b}$$

$$m_{2,2}(x') = \begin{cases} 0 & \text{if } f_2'(x') > -0.2622 \\ \dfrac{-0.2622 - f_2'(x')}{-0.2622 + 0.7338} & \text{if } -0.7338 < f_2'(x') \le -0.2622 \\ 1 & \text{if } f_2'(x') \le -0.7338 \end{cases} \tag{63a}$$

$$m_{2,2}(X) = 1 - m_{2,2}(x') \tag{63b}$$

The overall belief structures, $[m_i(x'), m_i(\neg x'), m_i(X) : i = 1, 2]$, are determined by merging the original belief structures using Eqs. (29–31). The satisfaction function is defined as

$$S_f(x') = S_2(x') \tag{64}$$

where $S_2(x')$ is evaluated using the recursive formulas given by Eqs. (33–36). The optimum solution of the problem is then found by maximizing the degree of satisfaction:

$$S_f(x'^*) = \max_{x' \in G'}[S_2(x')] \tag{65}$$

where the feasible domain G' is defined by Eq. (59). The optimum solution of the problem is given in the last column of Table 2. This solution represents the best compromise between the three design criteria and corresponds to a maximum level of satisfaction of 0.8631. Note that the original problem has fuzzy data and multiple objectives. There will be no unique solution that minimizes all the objective functions simultaneously. Also, the optimum solution of a crisp problem cannot be valid in the presence of fuzzy data. As such, the solution found using the present method can be considered the best compromise solution of the problem. When the problem was solved considering all the three objectives to be fuzzy (using the procedure of Ref. 7), the maximum level of satisfaction was found to be 0.691775.

Concluding Remarks

A generalized hybrid approach has been presented for solving fuzzy multicriterion optimization problems. A new fuzzy model of multiobjective optimization is established for engineering systems in which some of the objectives and/or constraints are ambiguous. The DS theory is modified to make it applicable for numerical optimization so that incomplete computational information, acquired during the iterative process of optimization, can be incorporated comprehensively. The fuzzy multiobjective optimization problem is ultimately transformed into a nonfuzzy (crisp) single-objective optimization problem so that ordinary mathematical programming techniques can be applied for its solution. Although, for simplicity, linear-shaped relationships have been used to represent both the membership and the bpa functions, the procedure is general and is applicable even with nonlinear relationships (such as exponential, hyperbolic, quadratic, logarithmic, and sine types[8]). The selection of a specific shape for membership functions might be based on the available real-world or empirical data, and might have an effect on the solution of the problem. Although the methodology was illustrated with a structural design example, the procedure is quite general and is applicable to any multicriterion optimization problem.

Appendix A: Basic Concepts of Fuzzy Set Theory

Let X be a classical (crisp) set of objects, called the universe, whose generic elements are denoted x. Membership in a classical subset A of X can be viewed as a characteristic function μ_A from X to $\{0, 1\}$ such that

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{A1}$$

The set $\{0, 1\}$ is called a valuation set. A set A is called a fuzzy set if the valuation set is allowed to be the real interval $[0, 1]$. The fuzzy set A is completely characterized by the set of pairs

$$A = \{[x, \mu_A(x)], x \in X\} \tag{A2}$$

where $\mu_A(x)$ is called the grade-of-membership function or degree of compatibility of x in A. The closer the value of $\mu_A(x)$ is to 1, the more x belongs to A. Clearly, A is a subset of X that has no sharp boundary. When X is a finite set $\{x_1, \ldots, x_n\}$, a fuzzy set on X can also be expressed as

$$A = \mu_A(x_1)|_{x_1} + \mu_A(x_2)|_{x_2} + \cdots + \mu_A(x_n)|_{x_n} = \sum_{i=1}^{n} \mu_A(x_1)|_{x_i} \tag{A3}$$

When X is continuous, A can be expressed as

$$A = \int_x \mu_A(x)|_x \tag{A4}$$

When two fuzzy sets A and B, with the corresponding supports for membership $\mu_A(y)$ and $\mu_B(y)$, are given, the fuzzy union of the sets is defined as

$$\mu_{A\cup B}(y) = \mu_A(y) \vee \mu_B(y) = \max[\mu_A(y), \mu_B(y)] = \begin{cases} \mu_A(y) & \text{if } \mu_A > \mu_B \\ \mu_B(y) & \text{if } \mu_A < \mu_B \end{cases} \tag{A5}$$

The fuzzy intersection is defined as

$$\mu_{A\cap B}(y) = \mu_A(y) \wedge \mu_B(y) = \min[\mu_A(y), \mu_B(y)] = \begin{cases} \mu_A(y) & \text{if } \mu_A < \mu_B \\ \mu_B(y) & \text{if } \mu_A > \mu_B \end{cases} \tag{A6}$$

The complement of a fuzzy set A, denoted as $\bar{A}$ and is given by $\mu_{\bar{A}}(y) = 1 - \mu_A(y)$.

In a fuzzy environment the conventional notion of optimization is revised. The fuzzy objective function is characterized by its membership function, and so are the constraints. The optimization (decision) in a fuzzy environment is to be viewed as the intersection of fuzzy constraints and fuzzy objective function. If $\mu_f(x)$ and $\mu_c(x)$ denote the membership functions of the objective and constraint functions, respectively, the optimal design or decision is characterized by the membership function

$$\mu_D(x) = \mu_f(x) \wedge \mu_c(x) \tag{A7}$$

Appendix B: Basic Concepts of DS Theory

The DS theory provides a means for representing situations in which different kinds of ignorance exist in our knowledge about a phenomenon or object. There are three important concepts that constitute the framework of DS theory as outlined below.

Basic Probability Assignment

Let X be a finite set whose elements x_i are called elementary propositions, each of which can be true or false. The universe of discourse is defined as an exhaustive set of all elementary propositions of our interest that are mutually exclusive. The notation 2^X is used to indicate the power set of X, which comprises all possible propositions of our interest; a proposition that is always true is symbolized by the universe of discourse $X \in 2^X$, and a proposition that is always false is symbolized by the empty set $\emptyset \in 2^X$. A function

$$m : 2^X \to [0, 1] \tag{B1}$$

is assigned to each element of 2^X to numerically expressed our degree of belief in a proposition. Thus, $m(A)$ represents the portion (possibility) of total belief assigned to the proposition A. The measure $m(A)$ is called the bpa function defined on the power set 2^X of the universe of discourse X. It satisfies the following axioms[4]:

$$0 \leq m(A) \leq 1 \qquad \text{for any} \quad A \in 2^X \tag{B2}$$

$$m(\emptyset) = 0 \tag{B3}$$

($\emptyset$ = empty set), and

$$\sum_{A\in 2^X} m(A) = 1 \tag{B4}$$

The basic probability assignment function appears to be similar to a probability measure; however, there exist some differences between the two[4]:

1) Additivity is not necessarily satisfied: in general,

$$m(\{x\}) + m(\{y\}) \neq m(\{x, y\}) = m(X) \tag{B5}$$

where $\{x\} \cap \{y\} = \emptyset$. A probability measure $P(A)$ on the other hand, satisfies the additivity relation:

$$P(A_i \cup A_j) = P(A_i) + P(A_j) \qquad \text{if } A_i \cap A_j = \emptyset \quad \text{for } i \neq j \tag{B6}$$

2) Monotonicity is not necessarily satisfied: in general,

$$m(\{y\}) \geq m(X) \qquad \text{even if} \quad \{y\} \subset X \tag{B7}$$

A probability measure $P(A)$ on the other hand, satisfies the monotonicity relation:

$$P(A) \leq P(B) \qquad \text{if } A \subseteq B \tag{B8}$$

3) The equation $m(\emptyset) = 0$ does not imply $m(X) = 1$ in the DS theory; $m(\{x\})$ may not be equal to $1 - m(\{\neg x\})$, where $\{\neg x\}$ represents the negation of the proposition $\{x\}$. On the other hand, in probability theory, $P(A) = 1$ implies $P(\emptyset) = 0$ and vice versa.

4) $m(X)$ need not be equal to 1, one may have $m(X) \leq 1$.

Belief Structure

A belief structure is defined as a collection of numbers $m(A)$ for $A \in 2^X$ that is expressed as

$$[m(A) : A \in 2^X] \tag{B9}$$

In this work, a belief structure called a simple support function on proposition A is used. This belief structure satisfies the relation

$$m(A) + m(X) = 1 \tag{B10}$$

Note that Eq. (B10) is also applicable to the proposition $\neg A$, where $m(\neg A)$ is known as the complementary belief structure. A simple support function conveys the information that the proposition A is believed with a numerical degree of support $m(A)$, called its degree of belief, and any proposition that is distinct from A in neither belief nor disbelief to the extent given by $m(X)$, called the degree of ignorance.

Combining Belief Structures

One of the key issues in DS theory deals with the method of incorporating the belief structures provided by multiple independent knowledge sources, each of which constitutes a professional reasoning or intelligent sensing. Two well-known rules of combination, Dempster's rule and Yager's rule, are briefly described.

Assume that there exist two independent knowledge sources pertaining to the belief structures $\{m_1(A){:}A \in 2^X\}$ and $\{m_2(B){:}B \in 2^X\}$, respectively, on the same universe of discourse X. Let the symbol C denote a new proposition merged from A and B. Then Dempster's rule computes $m(C)$ as

$$m(C) = \begin{cases} \dfrac{\sum_{A\cap B=C} m_1(A)m_2(B)}{1 - \sum_{A\cap B=\emptyset} m_1(A)m_2(B)} & \text{if } C \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{B11}$$

Yager's rule for evaluating $m(C)$ can be expressed as

$$m(C) = \begin{cases} \displaystyle\sum_{A\cap B=C} m_1(A)m_2(B) & \text{if } C \neq X, \emptyset \\ \displaystyle\sum_{A\cap B=\emptyset} m_1(\emptyset)m_2(\emptyset) + \sum_{A\cap B=X} m_1(X)m_2(X) & \text{otherwise} \end{cases} \tag{B12}$$

Dempster's rule is commonly used in cases where the independent knowledge sources are nonconflicting, whereas Yager's rule is applicable when they are mutually conflicting.

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